Mean-field theory of quantum Brownian motion

A.E. Allahverdyan[1,3,a], R. Balian[1], and Th.M. Nieuwenhuizen[2]

[1] CEA/Saclay, Service de Physique Théorique, 91191 Gif-sur-Yvette Cedex, France
[2] Department of Physics and Astronomy, University of Amsterdam, Valckenierstraat 65,
   1018 XE Amsterdam, The Netherlands
[3] Yerevan Physics Institute, Alikhanian Brothers St. 2, Yerevan 375036, Armenia

Abstract. We investigate a mean-field approach to a quantum Brownian particle interacting with a quantum thermal bath at temperature T, and subjected to a non-linear potential. An exact, partially classical description of quantum Brownian motion is proposed, which uses negative probabilities in its intermediate steps. It is shown that properties of the quantum particle can be mapped to those of two classical Brownian particles in a common potential, where one of them interacts with the quantum bath, whereas another one interacts with a classical bath at zero temperature. Due to damping the system allows a unique and non-singular classical limit at $\hbar \to 0$. For high T the stationary state becomes explicitly classical. The low-temperature case is studied through an effective Fokker-Planck equation. Non-trivial purely quantum correlation effects between the two particles are found.

PACS. 05.70.Ln Nonequilibrium and irreversible thermodynamics – 05.10.Gg Stochastic analysis (Fokker-Planck, Langevin) – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

The main conceptual problem in quantum mechanics remains the link between the quantum and the classical worlds. Therefore, significant efforts were made over years to understand at least part of the quantum world in classical notions. Curiously, the quasiclassical domain, which should be the main subject of this understanding, is still itself under extensive investigation. Indeed, it is known to be non-trivial; in a sense it can be even more complex that the classical and quantum extremes alone. It is important to realize in this context that the quasiclassical domain is not exhausted by the conventional ansatz $\hbar \to 0$ [1–3], since this limit is singular (therefore some *a priori* concepts similar to coarse-graining are sometimes involved [3]), and since it does not commute with other limits of physical interest, *e.g.* the limit of large times.

One of the established approaches to the quasiclassical domain is a collection of mean-field methods known as Gaussian decoupling procedure or time-dependent variational approximation [4–7]. Mean-field (variational, Hartree-Fock) methods are well-known in the quantum theory, and were applied for a while in many different areas. This set of methods appeared to be especially suitable for the quasiclassical domain, since it attempts to realize in a simple and straightforward way the above-mentioned program of understanding the quantum theory

in classical terms. An impressive amount of experimental confirmations, in particular in quantum chemistry and atomic physics [4], numerical and self-consistency checks were made for those methods. Therefore, they have already become a well-formulated and sound approach.

In the present paper we apply this mean-field method to the simplest quantum dissipative system: A quantum Brownian particle interacting with a thermal bath. There are specific reasons to study this type of quantum systems in the context of the above-mentioned problems. In contrast to closed Hamiltonian systems, quantum dissipative systems are more reliable candidates to understand their physics in classical terms. Indeed, a non-unitary evolution provides a natural mechanism of decoherence [14], and the need of artificial coarse-graining procedures is gotten rid of. In the light of this conceptual advantage it should be surprising that the basic understanding of their theory is still rather fragmentary. This is so mainly because one can use neither general properties of unitary evolution, which describes closed systems, nor Markovian properties of the classical stochastic dynamics, since due to the correlation time $\hbar/T$, which is relevant at low temperatures, the corresponding statistical dynamics is essentially non-Markovian. Each of these properties has important general consequences, which are, thus, absent in the quantum dissipative case [9]. The only exceptions are weakly damped high-temperature systems, where an influence of the bath is described classically, whereas the rest remains quantum-mechanical [14]. We will not be concerned with

[a] e-mail: `aarmen@moon.yerphi.am`

them within the present paper. A fairly general theory of strongly-damped and/or low-temperature dissipative systems is still under construction, though some suggestive results were obtained recently [15], where for a class of systems two of us proposed a consistent statistical thermodynamical theory of quantum Brownian motion.

Our plan is the following. In Section 2 we will first briefly recall some known facts on quantum Langevin equations and quantum noise. Here we obtain also generalized Wigner-Moyal and von Neumann equations, which are exact consequences of the quantum dynamics, but describe the Brownian particle in almost classical terms. The Gaussian approximation will be presented in Section 3, where we will obtain the basic equations of the present paper, and draw some general conclusions. Here we will discuss, in particular, what is the constructive role of friction when establishing quantum-classical transition. The effective Fokker-Planck dynamics will be discussed in Section 4. In Section 5 we study low-temperature properties of the model. We conclude in the last section.

2 Quantum Langevin equation and Liouville-Moyal equation

2.1 Quantum Langevin equation

This fundamental equation of quantum Brownian motion theory is derived from the exact Hamiltonian description of a subsystem (Brownian particle) and a thermal bath, when tracing out the degrees of freedom of the bath. The standard assumption is that at the moment $t = 0$ the states of the subsystem and the bath were decoupled from each other, and the bath was in equilibrium at temperature T [9,10,15]. Further, the influence of the particle to the bath is assumed to be sufficiently small; thus, only the linear modes of the bath are excited, and the interaction between the particle and the bath is linear. Since the dynamics of the bath is linear, it can be solved exactly. Following this line of exact calculations [9], one derives the quantum Langevin equation

$$
\dot{\hat{x}} = \frac{\hat{p}}{m},
$$
$$
\dot{\hat{p}} + \frac{1}{m}\int_0^t \mathrm{d}t' G(t-t')\hat{p}(t') + V'(\hat{x})
$$
$$
= -\gamma\Gamma e^{-\Gamma t}\hat{x}(0) + \hat{\eta}(t), \tag{1}
$$

where $\hat{p}(t)$ $\hat{x}(t)$ are Heisenberg operators of momentum and coordinate, and $V(x)$ is an external potential. The parameter γ is the damping constant, which determines the interaction between the bath and the particle. For $\gamma \to 0$ one gets from equation (1) the usual Heisenberg equations. Γ is the maximal characteristic frequency of the bath, and it determines the retardation time of the friction kernel

$$
G(t) = \gamma\Gamma e^{-\Gamma|t|}. \tag{2}
$$

The operator $\hat{\eta}(t)$ is the random noise, which appeared due to the uncertain character of the initial (equilibrium) distribution of the bath. This noise can be shown to be Gaussian, due to the fact that the thermal bath is a harmonic system and was in equilibrium. It has the following properties:

$$
K(t) = \frac{1}{2}\langle\hat{\eta}(t)\hat{\eta}(0) + \hat{\eta}(0)\hat{\eta}(t)\rangle_{\hat{\eta}} \equiv \frac{1}{2}\langle\hat{\eta}(t);\hat{\eta}(0)\rangle_{\hat{\eta}}
$$
$$
= \frac{\hbar\gamma}{\pi}\int_0^\infty \mathrm{d}\omega\,\omega\coth\left(\frac{\hbar\omega\beta}{2}\right)\cos(\omega t)\frac{1}{1+(\omega/\Gamma)^2}, \tag{3}
$$

$$
\hat{\eta}(t)\hat{\eta}(0) - \hat{\eta}(0)\hat{\eta}(t) \equiv [\hat{\eta}(t),\hat{\eta}(0)] = \mathrm{i}\hbar\frac{\partial\gamma(t)}{\partial t}. \tag{4}
$$

Hereafter we use

$$
\langle\hat{A};\hat{B}\rangle \equiv \frac{1}{2}\langle\hat{A}\hat{B} + \hat{B}\hat{A}\rangle, \quad [\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A},
$$
$$
[\hat{A},\hat{B}]_+ \equiv \hat{A}\hat{B} + \hat{B}\hat{A} \tag{5}
$$

for any operators $\hat{A}$, $\hat{B}$.

The connection between properties of the noise and the friction kernel is the consequence of quantum fluctuation-dissipation theorem [9]. Equation (1) with physically suitable forms of the potential and friction describes a rich variety of physical phenomena (see Refs. in [9,10,15]).

2.1.1 Quasi-Ohmic limit

In the present paper we shall restrict ourselves to the *quasi-Ohmic* case [8], where Γ is much larger than other characteristic times, but still finite. The main reason of this approximation is to have an exact equation for the Wigner function of the Brownian particle, which will be derived in the next section.

For the quantum noise one has that $K(t) = -\ln(\Gamma t) > 0$ for small times, and for $t \gg 1/\Gamma$, $K(t)$ is anticorrelated with the universal correlation time $\hbar\beta/2\pi$:

$$
K(t) = -\frac{\pi\gamma T^2}{\hbar}\left[\sinh\left(\frac{\pi t}{\beta\hbar}\right)\right]^{-2}, \tag{6}
$$

Being coherent, the low-temperature quantum thermal bath necessarily generates a colored noise. The classical white noise situation is recovered when taking the high-temperature limit ($\hbar\beta \to 0$). In general, this should be done before the limit $\Gamma \to \infty$. Notice that in contrast with the classical case, the quantum noise does not disappear for $T \to 0$, since even in this limit the initial state of the quantum thermal bath remains indeterminate.

In the quasi-Ohmic regime one can expand the memory kernel for the friction in equation (1) as:

$$
\int_0^t \mathrm{d}t' e^{-\Gamma(t-t')}\hat{p}(t') = \frac{1}{\Gamma}\hat{p}(t) - \frac{1}{\Gamma^2}\dot{\hat{p}}(t)
$$
$$
- \frac{1}{\Gamma}e^{-\Gamma t}\hat{p}(0) + \frac{1}{\Gamma^2}e^{-\Gamma t}\dot{\hat{p}}(0). \tag{7}
$$

For $t > 0$ and large Γ the exponentially small factors depending on the initial conditions can be omitted in equations (1, 7), and we finally get

$$\dot{p} + \frac{\gamma}{m}\hat{p}(t) + V'(\hat{x}) = \hat{\eta}(t), \qquad (8)$$

In other words, for not very small times one can regularize only the noise, but keep the friction local, when considering the quasi Ohmic case.

Let us finally notice the following general commutation relation between the noise and an arbitrary operator $\hat{A}(t)$ of the particle [9,11]:

$$[\hat{A}(t), \hat{\eta}(s)] = \int_0^t dt' [\hat{A}(t), \hat{x}(t')] \frac{d}{ds} \gamma(s - t'). \qquad (9)$$

Here the coordinate operator appeared just because the interaction with the thermal bath is taking place through the coordinate [11,9]. In the quasi-Ohmic case this will go to

$$[\hat{A}(t), \hat{\eta}(s)] = 2\gamma \frac{d}{ds} \{\theta(t - s)[\hat{A}(t), \hat{x}(s)]\}, \qquad (10)$$

where $\theta(t - s)$ is the step-function ($\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$) defined to be $1/2$ at $t = s$. It is seen that $[\hat{\eta}(t), \hat{x}(t)] = [\hat{\eta}(t), \hat{p}(t)] = 0$. One could attribute these relations to causality, but they are an emergent property of the quasi-Ohmic limit.

2.2 Generalized Wigner-Moyal equation

The quantum Langevin equation (8) is a non-linear operator equation, and as such it can hardly be handled directly. Here we will present a generalized Wigner-Moyal equation for the corresponding Wigner function of the particle, which exactly corresponds to equation (8) in the same sense as the pure Heisenberg equations correspond to the usual Wigner-Moyal equation. Besides technical advantages which will be used further, this equation presents an interesting account for an exact description of the non-linear quantum problem through proper classical terms.

One is looking for an equation for the Wigner function:

$$W(p, x, t) = \langle \text{tr}\rho_0 \ \hat{W}(\hat{p}, \hat{x}, t)\rangle_{\hat{\eta}},$$
$$\hat{W}(\hat{p}, \hat{x}, t) = \int \frac{da\,db}{4\pi^2} \exp(-iax - ibp + ia\hat{x}(t) + ib\hat{p}(t)), \qquad (11)$$

where ρ_0 is the initial (at the moment $t = 0$) density matrix of the Brownian particle. $\hat{W}(\hat{p}, \hat{x}, t)$ is the support of the Wigner function, which can be viewed as a quantum analogue of the delta-function. In the classical limit, where $\hat{x}$ and $\hat{p}$ approximately commute, $W(p, x, t)$ tends to the ordinary probability distribution of the coordinate and momentum.

Since the equation for $W(p, x, t)$ is expected to be linear, one is interested by an equation for $\hat{W}(\hat{p}, \hat{x}, t)$,

whereas the averages can be taken later. The derivation of this equation is fairly straightforward, since it uses equations (8, 11) and the standard commutation relation $[\hat{x}(t), \hat{p}(t)] = i\hbar$. We will write only the final result:

$$\frac{\partial \hat{W}(\hat{p}, \hat{x}, t)}{\partial t} = -\frac{\partial}{\partial x}(\frac{p}{m}\hat{W}) + \frac{\partial}{\partial p}([V'(x) + \frac{\gamma}{m}p]\hat{W})$$
$$+ \sum_{n=1}^{\infty} \frac{(i\hbar/2)^{2n}}{(2n+1)!} \frac{\partial^{2n+1}V}{\partial x^{2n+1}} \frac{\partial^{2n+1}\hat{W}}{\partial p^{2n+1}} - \frac{\partial}{\partial p}(\hat{\eta}\hat{W}). \quad (12)$$

The first three terms in the r.h.s of this equation are the standard drift terms of the classical Liouville equation, and the sum represents a purely quantum correction, which comes from the non-linearity of the potential.

Our object of interest is the last term, which upon averaging will look like

$$\frac{\partial}{\partial p}\langle\hat{\eta}(t)\hat{W}(t)\rangle_{\hat{\eta}}. \qquad (13)$$

Notice that $\hat{\eta}(t), \hat{W}(t)$ commute, since $\hat{\eta}(t)$ commutes with $x(t), p(t)$, and $\hat{W}(t)$ can be regarded as a product of two terms, which depend only on $\hat{x}(t)$ and $\hat{p}(t)$ correspondingly.

Let us adopt the following formal expansion for $\hat{W}(t)$

$$\hat{W}(t|\eta) = \hat{W}(t|0)$$
$$+ \frac{1}{n!}\sum_{n=1}^{\infty} \int_0^t ds_1...ds_n R(s_1,..,s_n) \Pi[\hat{\eta}(s_1)...\hat{\eta}(s_n)], \quad (14)$$

$$\Pi[\hat{\eta}(s_1)...\hat{\eta}(s_n)] = \frac{1}{n!}\sum_{i_1 \neq ... \neq i_n} \hat{\eta}(s_{i_1})...\hat{\eta}(s_{i_n}). \qquad (15)$$

Here $\Pi[\hat{\eta}(s_1)...\hat{\eta}(s_n)]$ is the symmetrized product; the coefficients $R(s_1,..,s_n)$ are c-numbers. Due to the fact that $[\hat{W}(t), \hat{\eta}(t)] = 0$ one has:

$$\langle\hat{\eta}(t)\hat{W}(t)\rangle_{\hat{\eta}} = \langle\hat{\eta}(t); \hat{W}(t)\rangle_{\hat{\eta}} =$$
$$\sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t ds_1...ds_n R(s_1,..,s_n)\langle\hat{\eta}(t); \Pi[\hat{\eta}(s_1)...\hat{\eta}(s_n)]\rangle_{\hat{\eta}}. \qquad (16)$$

Since $\hat{\eta}(t)$ is a Gaussian random operator with $\langle\hat{\eta}(t)\rangle_{\hat{\eta}} = 0$, one can use Wick's theorem: The correlation of an odd number of $\hat{\eta}$'s vanishes. The correlation of an even number of $\hat{\eta}$'s is equal to the sum of products of pair correlations, the sum being taken over all pairings. For example:

$$\langle\hat{\eta}(t_1)\hat{\eta}(t_2)\hat{\eta}(t_3)\hat{\eta}(t_4)\rangle_{\hat{\eta}} = \langle\hat{\eta}(t_1)\hat{\eta}(t_2)\rangle_{\hat{\eta}}\langle\hat{\eta}(t_3)\hat{\eta}(t_4)\rangle_{\hat{\eta}}$$
$$+ \langle\hat{\eta}(t_1)\hat{\eta}(t_3)\rangle_{\hat{\eta}}\langle\hat{\eta}(t_2)\hat{\eta}(t_4)\rangle_{\hat{\eta}}$$
$$+ \langle\hat{\eta}(t_1)\hat{\eta}(t_4)\rangle_{\hat{\eta}}\langle\hat{\eta}(t_2)\hat{\eta}(t_3)\rangle_{\hat{\eta}}. \qquad (17)$$

In this way one derives:

$$\langle\hat{\eta}(t); \Pi[\hat{\eta}(s_1)...\hat{\eta}(s_n)]\rangle_{\hat{\eta}} =$$
$$\sum_{\alpha=1}^{n} \langle\hat{\eta}(t); \hat{\eta}(s_\alpha)\rangle_{\hat{\eta}} \ \langle\Pi[\hat{\eta}(s_1)...\hat{\eta}(s_{\alpha-1})\hat{\eta}(s_{\alpha+1})...\hat{\eta}(s_n)]\rangle_{\hat{\eta}}. \qquad (18)$$

Having substituted this equation to equation (16) one gets

$$\langle \hat{\eta}(t)\hat{W}(t)\rangle_{\hat{\eta}} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_0^t \mathrm{d}s_1 \langle \hat{\eta}(t); \hat{\eta}(s_1)\rangle_{\hat{\eta}}$$

$$\times \int_0^t \mathrm{d}s_2...\mathrm{d}s_n R(s_1,..,s_n)\langle \Pi[\hat{\eta}(s_2)...\hat{\eta}(s_n)]\rangle_{\hat{\eta}}. \quad (19)$$

Now one notices that the only feature of the quantum noise which enters here is the autocorrelation function $\langle \hat{\eta}(t); \hat{\eta}(s)\rangle_{\hat{\eta}} = K(t-s)$ because in the end $\langle \Pi[\hat{\eta}(s_2)...\hat{\eta}(s_n)]\rangle_{\hat{\eta}}$ can be expressed through it using equation (18) several times. So nothing will change if we replace $\hat{\eta}(t)$ in equation (19) by a *classical* Gaussian noise $\eta(t)$ which has the same autocorrelation function $K(t-s)$,

$$\langle \hat{\eta}(t)\hat{W}(t)\rangle_{\hat{\eta}} = \langle \eta(t)\hat{W}(t)\rangle_{\eta}. \quad (20)$$

Now we substitute this result into equations (12), take the average over the initial state, but take out the average over the classical noise:

$$\partial_t w = -\partial_x(\frac{p}{m}w) + \partial_p([V'(x) + \frac{\gamma}{m}p - \eta(t)]w)$$

$$+ \sum_{n=1}^{\infty} \frac{(\mathrm{i}\hbar/2)^{2n}}{(2n+1)!} \frac{\partial^{2n+1}V}{\partial x^{2n+1}} \frac{\partial^{2n+1}w}{\partial p^{2n+1}}, \quad (21)$$

where the true Wigner function $W(p,x,t)$ will be obtained by averaging the auxiliary object $w(p,x,t)$ over the classical noise $\eta(t)$:

$$W(p,x,t) = \langle w(p,x,t)\rangle_{\eta}. \quad (22)$$

Thus in the quasi-Ohmic limit one can treat the quantum noise as a purely classical object as far as the Wigner function is concerned. Notice that this result is exact and relies only on the quasi-Ohmic limit. In that respect it is different from the description through semiclassical Langevin equations, where in the overdamped (large γ) limit one also gets an analogue of equation (21) but without the $\hbar$-dependent terms.

2.3 Generalized von Neumann equation

Here we will investigate the von Neumann equation for a matrix $r(x,x',t)$, which corresponds to the auxiliary Wigner function $w(p,x,t)$:

$$w(p,x,t) = \frac{1}{2\pi\hbar} \int \mathrm{d}u \, r(x-\frac{u}{2}, x+\frac{u}{2}, t) \, \mathrm{e}^{\mathrm{i}p\,u/\hbar}. \quad (23)$$

Since equation (21) is linear, one directly obtains for $r(x,x',t) = \langle x'|\hat{r}(t)|x\rangle$

$$-\mathrm{i}\hbar\frac{\partial}{\partial t}r(x,x',t) = \left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2} - V(x) + V(x')\right.$$

$$\left. +\eta(t)(x-x') + \frac{\mathrm{i}\gamma\hbar}{m}(x-x')(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})\right] r(x,x',t), \quad (24)$$

or in the equivalent operator notations:

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[\hat{H} - \hat{x}\eta, \hat{r}] + \frac{\mathrm{i}\gamma}{2\hbar m}[\{\hat{p}, \hat{r}\}, \hat{x}]. \quad (25)$$

The true density matrix $\hat{\rho}$ can be obtained after averaging over the classical noise $\eta(t)$:

$$\hat{\rho}(t) = \langle \hat{r}(t)\rangle_{\eta}. \quad (26)$$

Therefore, all possible averages are obtained as

$$\mathrm{tr}(\hat{\rho}\hat{A}) = \langle \mathrm{tr}(\hat{r}\hat{A})\rangle_{\eta}, \quad (27)$$

where an operator $\hat{A}$ lives in the Hilbert space of the Brownian particle.

It should be stressed that $\hat{r}$ is not a density matrix itself, but rather a tool to calculate averages. Indeed starting from equation (25) one easily gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\,\hat{r} = 0 \rightarrow \mathrm{tr}\,\hat{r}(t) = 1, \quad (28)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(\hat{r}(t)^2) = \frac{\gamma}{m}\mathrm{tr}(\hat{r}(t)^2) \rightarrow \mathrm{tr}(\hat{r}(t)^2) = \mathrm{e}^{\gamma t/m}\mathrm{tr}(\hat{r}(0)^2). \quad (29)$$

These indicate that r has negative eigenvalues for $t > 0$, the absolute value of which grows with time. However, positive eigenvalues compensate this growth in such a way that equation (28) holds. In particular, equations (28, 29) show that a dissipative system cannot be described by a wave function even if the averaging over the noise is postponed.

Thus we see that the "classicalization" of the noise is not just a technical procedure, but it has to be accompanied with a change of interpretation: The unaveraged density matrix $\hat{r}$ is not a true density matrix, since it does have negative eigenvalues. In other words, explicitly classical components of the dynamics lead to the appearance of negative probabilities. On the other hand, there are no reasons to consider those negative eigenvalues as something unphysical: Our derivation of equations (21, 25) was exact, and later we will present other indications that neither the true density matrix, nor averages calculated according to equation (27) show unphysical properties (see Eq. (53), and the discussion after (56)). The situation is the same as for the Wigner function at a given time, which is not a positive probability density, but can be safely used to evaluate expectation values by integration.

The appearance of negative probabilities as a result of imposing partially classical properties has certain analogies with the classical interpretation of quantum entanglement, and in particular Einstein-Podolsky-Rosen phenomenon. This interpretation also uses negative (though not directly observable) probabilities [12].

Finally we would like to mention that the reported non-positive character of the unaveraged density matrix has nothing to do with the known technical problem which also appears through non-positive (averaged) density matrices in certain quantum Markovian diffusion equations [13]. There the problem is merely technical and

arises due to the fact that the Markovian approximation in the quantum theory of open systems is essentially time-inhomogeneous, so that its careless use leads to such problems. Indeed, the problem disappears after a more consistent treatment of the situation [13]. In our case, in contrast, the non-positive character of the unaveraged density matrix is an exact consequence of our attempt to handle the quantum noise classically.

3 Gaussian decoupling procedure

Equation (21) is, of course, intractable in general. It joins all technical difficulties of the classical Liouville equation with a given noise and friction and those of the pure Moyal equation. In other words some substantial simplifications are necessary to proceed further. Here we will apply the mean-field approach, namely, a solution of equation (21) will be looked through its moments, and the Gaussian decoupling procedure will be applied to the higher-order moments. By its spirit this is very similar to the Grad method in the kinetic theory of rarefied gases [17], and has been applied recently in quantum theory of closed systems as well [4,6]. It is clear that the consistency of this approximation should be checked together with the final solution. At the moment we will notice only that since its application is connected with the weakness of quantum fluctuations, it will have a reliable range of validity in the quasiclassical domain.

The working variables will be

$$d_x = \langle \hat{x} \rangle = \int \mathrm{d}p \, \mathrm{d}x \, x \, w(p, x, t), \tag{30}$$

$$d_p = \langle \hat{p} \rangle = \int \mathrm{d}p \, \mathrm{d}x \, p \, w(p, x, t), \tag{31}$$

$$d_{xx} = \langle (\hat{x} - \langle \hat{x} \rangle)^2 \rangle = \int \mathrm{d}p \, \mathrm{d}x \, (x - \langle x \rangle)^2 w(p, x, t), \tag{32}$$

$$d_{pp} = \langle (\hat{p} - \langle \hat{p} \rangle)^2 \rangle = \int \mathrm{d}p \, \mathrm{d}x \, (x - \langle x \rangle)^2 w(p, x, t), \tag{33}$$

$$d_{xp} = \frac{1}{2} \langle (\hat{x} - \langle \hat{x} \rangle)(\hat{p} - \langle \hat{p} \rangle) + (\hat{p} - \langle \hat{p} \rangle)(\hat{x} - \langle \hat{x} \rangle) \rangle$$
$$= \int \mathrm{d}p \, \mathrm{d}x \, (x - \langle x \rangle)(p - \langle p \rangle) w(p, x, t). \tag{34}$$

For the higher-order correlations one assumes the Gaussian decoupling:

$$\int \mathrm{d}p \, \mathrm{d}x \, w(p, x, t)(x - \langle x \rangle)^{2n} =$$
$$(2n-1)!! \left[\int \mathrm{d}p \, \mathrm{d}x \, w(p, x, t)(x - \langle x \rangle)^2 \right]^n, \tag{35}$$

$$\int \mathrm{d}p \, \mathrm{d}x \, w(p, x, t)(p - \langle p \rangle)(x - \langle x \rangle)^{2n+1}$$
$$= (2n+1)!! \left[\int \mathrm{d}p \, \mathrm{d}x \, w(p, x, t)(p - \langle p \rangle) \right]$$
$$\times \left[\int \mathrm{d}p \, \mathrm{d}x \, w(p, x, t)(x - \langle x \rangle)^2 \right]^n. \tag{36}$$

This just means that $w(p, x, t)$ is restricted to the subspace of Gaussian functions:

$$w(p, x, t | d_{xx}, d_{xp}, d_{pp}, d_x, d_p) = \frac{1}{2\pi\sqrt{\Delta}}$$
$$\times \exp\left(-\frac{1}{2\Delta}[d_{pp}(x - d_x)^2 + d_{xx}(p - d_p)^2 \right.$$
$$\left. - 2d_{xp}(x - d_x)(p - d_p)] \right),$$
$$\Delta = d_{xx}d_{pp} - d_{xp}^2. \tag{37}$$

Applying equations (30–36) in equation (21) one gets the following equations of motions, which are now classical equations for classical variables:

$$\frac{\mathrm{d}}{\mathrm{d}t}d_x = \frac{d_p}{m},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}d_p = -\frac{\gamma}{m}d_p + \eta(t) - V'(d_x) + \sum_{n=1}^{\infty} \frac{V^{(2n+1)}(d_x)}{(2n)!!}d_{xx}^n,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}d_{xx} = \frac{2}{m}d_{xp},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}d_{pp} = -\frac{2\gamma}{m}d_{pp} - 2\sum_{n=1}^{\infty} \frac{V^{(2n)}(d_x)}{(2n-2)!!}d_{xx}^{n-1}d_{xp},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}d_{xp} = -\frac{\gamma}{m}d_{xp} + \frac{1}{m}d_{pp} - \sum_{n=1}^{\infty} \frac{V^{(2n)}(d_x)}{(2n-2)!!}d_{xx}^n. \tag{38}$$

This set of equations can be considerably simplified if the following change of variables is made:

$$d_x = X, \tag{39}$$
$$d_p = P, \tag{40}$$
$$d_{xx} = Q^2, \tag{41}$$
$$d_{pp} = \Pi^2 + \frac{\hbar^2\sigma^2}{4Q^2}, \tag{42}$$
$$d_{xp} = Q\Pi, \tag{43}$$

where σ is chosen such that

$$\frac{1}{\sigma} = \mathrm{tr}(r^2). \tag{44}$$

With this change of variables equations (38) will read

$$\frac{\mathrm{d}}{\mathrm{d}t}X = \frac{P}{m}, \tag{45}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}P = -\frac{\gamma}{m}P + \eta(t) - \frac{\partial \mathcal{H}}{\partial X}, \tag{46}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}Q = \frac{\Pi}{m}, \tag{47}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi = -\frac{\gamma}{m}\Pi - \frac{\partial \mathcal{H}}{\partial Q}, \tag{48}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma = -\frac{\gamma}{m}\sigma \Rightarrow \sigma(t) = \mathrm{e}^{-\gamma t/m}\sigma(0) = \mathrm{e}^{-\gamma t/m}, \tag{49}$$

where the initial state was chosen to be pure for simplicity: $\sigma(0) = 1$, and where $\mathcal{H}$ is an effective classical Hamiltonian:

$$\mathcal{H}(P, X, \Pi, Q, t) = \frac{P^2}{2m} + \frac{\Pi^2}{2m} + V(X)$$
$$+ \sum_{n=1}^{\infty} \frac{V^{(2n)}(X)}{(2n)!!} Q^{2n} + \frac{\hbar^2 \sigma(t)^2}{8mQ^2}. \quad (50)$$

The true Wigner function for the original quantum particle reads:

$$W(p, x, t) =$$
$$\int \mathrm{d}P \, \mathrm{d}X \, \mathrm{d}\Pi \, \mathrm{d}Q \, w(p, x, t \,|\, P, X, \Pi, Q) \, \mathcal{P}(P, X, \Pi, Q, t),$$
$$(51)$$

where $\mathcal{P}(P, X, \Pi, Q, t)$ is the ordinary probability distribution of the classical random variables P, X, Π and Q at the moment t.

The physical meaning of this approach is now clear. The Hamiltonian (50) corresponds to two classical particles with coordinates X, Q and momenta P, Π. Equations (45–48) show that X-particle couples to the quantum bath through the damping $\gamma P/m$ and noise $\eta(t)$. Although $\eta(t)$ is not an operator, its correlator is still given by the quantum spectrum $K(t)$. Q-particle interacts with a classical bath at zero temperature, since only in this case a classical particle is subjected to damping, but not to noise. The effective Hamiltonian $\mathcal{H}$ non-trivially couples these two particles. It is also time-dependent, though this dependence is quite simple.

Already the general form of the effective Hamiltonian leads us to the following important observation. As follows from the derivation of the result, the purely classical case corresponds to $\hbar \to 0$, $\Pi \to 0$, $q \to 0$. Without damping one will have $\sigma = \text{const.}$, which just reflects unitary evolution, where r^2 is conserved. Since Q, Π typically have order $\mathcal{O}(\hbar)$, all terms in equation (50) should disappear in the classical limit. For all terms besides the last one this disappearance is clear. This last term can make the classical limit non-unique or even singular. This phenomenon is well-known in quantum chaos [2,6]. Moreover, even fairly simple integrable systems can display singularities in the classical limit [3]. Now we observe that this dangerous term $\hbar^2\sigma(t)^2/(8mQ^2)$ disappears with the characteristic time m/γ, thereby ensuring the relatively straightforward classical limit in a damped system. Notice in this context that usually it is only the noise, which is believed to facilitate the classical limit, providing a mechanism for decoherence [14]. It is useful to mention that the Gaussian property of the quantum noise (see the discussion before Eq. (3)) was important in the derivation of the generalized Wigner-Moyal equation (21) and for the existence of the effective Hamiltonian (50).

Notice that the equality

$$d_{xx}d_{pp} - d_{xp}^2 = \frac{\hbar^2\sigma^2}{4}, \quad (52)$$

when σ given by equation (49) is less than one, does not indicate a breaking of the uncertainty relations, since $\hat{r}$ itself is not a density matrix. The correct uncertainty relation will read as

$$\left(\langle\langle \hat{x}^2 \rangle\rangle_\eta - \langle\langle \hat{x} \rangle\rangle_\eta^2 \right) \left(\langle\langle \hat{p}^2 \rangle\rangle_\eta - \langle\langle \hat{p} \rangle\rangle_\eta^2 \right)$$
$$- \frac{1}{4} \left(\langle\langle (\hat{x} - \langle\langle\hat{x}\rangle\rangle_\eta) (\hat{p} - \langle\langle\hat{p}\rangle\rangle_\eta) \right.$$
$$\left. + (\hat{p} - \langle\langle\hat{p}\rangle\rangle_\eta) (\hat{x} - \langle\langle\hat{x}\rangle\rangle_\eta) \rangle\rangle_\eta \right)^2 \geq \frac{\hbar^2}{4}, \quad (53)$$

where $\langle\langle ... \rangle\rangle_\eta$ is the complete average, namely the average over r (indicated with $\langle ... \rangle$), and over the classical noise (indicated with $\langle ... \rangle_\eta$).

These observable averages can be expressed as

$$\langle\langle \hat{x}^2 \rangle\rangle_\eta - \langle\langle \hat{x} \rangle\rangle_\eta^2 = \langle d_{xx} \rangle_\eta + \langle d_x^2 \rangle_\eta - \langle d_x \rangle_\eta^2 \quad (54)$$
$$\langle\langle \hat{p}^2 \rangle\rangle_\eta - \langle\langle \hat{p} \rangle\rangle_\eta^2 = \langle d_{pp} \rangle_\eta + \langle d_p^2 \rangle_\eta - \langle d_p \rangle_\eta^2 \quad (55)$$

$$\frac{1}{2} \left(\langle\langle (\hat{x} - \langle\langle\hat{x}\rangle\rangle_\eta) (\hat{p} - \langle\langle\hat{p}\rangle\rangle_\eta) \right.$$
$$\left. + (\hat{p} - \langle\langle\hat{p}\rangle\rangle_\eta) (\hat{x} - \langle\langle\hat{x}\rangle\rangle_\eta) \rangle\rangle_\eta \right)$$
$$= \langle d_{xp} \rangle_\eta + \langle d_x d_p \rangle_\eta - \langle d_x \rangle_\eta \langle d_p \rangle_\eta. \quad (56)$$

Recall the situation with the exactly solvable harmonic potential. Here d_{xx}, d_{xp}, d_{pp} tend to zero in the long-time limit, being decoupled from d_x, d_p. Then equation (53) is obviously satisfied.

Let us briefly mention another aspect of the proposed scheme, which can be interesting on general grounds. Two important length-scales are associated with any quantum system

$$L_\mathrm{c} = \sqrt{D_{xx}}, \quad (57)$$
$$L_\mathrm{q} = \sqrt{\frac{\hbar^2 D_{xx}}{4(D_{pp}D_{xx} - D_{xp}^2)}}, \quad (58)$$

where D_{xx}, D_{pp} are dispersions of the coordinate and momentum, and D_{xp} is the corresponding cross-correlation. These lengths quantify the quantum (L_q) and classical (L_c) aspects of the system, since due to uncertainty relation: $L_\mathrm{c}/L_\mathrm{q} \geq 1$, and the classical limit corresponds to $L_\mathrm{c}/L_\mathrm{q} \gg 1$.

The analogous lengths for our mean-field scheme read

$$\tilde{L}_\mathrm{c} = \sqrt{d_{xx}}, \quad \tilde{L}_\mathrm{q} = \sqrt{\frac{\hbar^2 d_{xx}}{4(d_{pp}d_{xx} - d_{xp}^2)}}, \quad (59)$$
$$\tilde{L}_\mathrm{c}/\tilde{L}_\mathrm{q} \sim \mathrm{e}^{-\gamma t/m}. \quad (60)$$

The lengths $\tilde{L}_\mathrm{c}$, $\tilde{L}_\mathrm{q}$ still characterize classical and quantum effects, but the uncertainty relation does not apply to them, due the above remarks. Ratio (60) can now be much smaller than one, namely the quantum effects can be overdominating. The noise is needed to recover the uncertainty relation and to limit the overspread of quantum effects.

To conclude this section we will notice that recently an attempt was made to construct a mean-field theory for a quantum Brownian particle [16]. However, the authors did not start from the correct statement of the problem, and were led to an incorrect result that their open quantum system can be still described by a wave function (*i.e.*, a pure state). In our notations this will amount to put $\sigma(t) = 1$ for all $t > 0$ which is clearly incorrect in the light of equations (49, 29). Besides the technical aspect it contradicts to the general qualitative statement which we draw after equation (49).

4 Fokker-Planck equation

Equations (45–48) are still fairly complicated non-linear equations. To study them especially at low temperatures we shall employ methods recently developed by two of us [15].

We are looking for an equation describing the common probability distribution

$$\mathcal{P}(y_1, y_2, y_3, y_4, t) = \\ \langle \delta(y_1 - P(t))\delta(y_2 - X(t))\delta(y_3 - \Pi(t))\delta(y_4 - Q(t))\rangle_\eta. \tag{61}$$

Using equations (45–48) and direct differentiation one will get

$$\frac{\partial \mathcal{P}}{\partial t} = \sum_{k=1}^{4} \frac{\partial(v_k P)}{\partial y_k} - \frac{\partial}{\partial y_1}\langle \delta(P(t) - y_1)\delta(X(t) - y_2) \\ \times \delta(\Pi(t) - y_3)\delta(Q(t) - y_4)\eta(t)\rangle_\eta, \tag{62}$$

where

$$v_1 = \frac{\gamma}{m}y_1(t) + \partial_{y_1}\mathcal{H}(y_1, y_2, y_3, y_4), \tag{63}$$

$$v_2 = -\frac{y_1}{m}, \tag{64}$$

$$v_3 = \frac{\gamma}{m}y_3(t) + \partial_{y_2}\mathcal{H}(y_1, y_2, y_3, y_4), \tag{65}$$

$$v_4 = -\frac{y_3}{m}. \tag{66}$$

Since the Gaussian noise is distributed with a functional

$$\Omega[\eta] \sim \exp -\frac{1}{2}\int \mathrm{d}t\mathrm{d}s\, \eta(t)K^{-1}(t-s)\eta(s), \tag{67}$$

one has

$$\eta(t)\Omega[\eta] = -\int \mathrm{d}t'K(t'-t)\frac{\delta\Omega[\eta]}{\delta\eta(t')}, \tag{68}$$

Substituting this equation into equation (62) one obtains after functional integration by parts:

$$\frac{\partial \mathcal{P}}{\partial t} = \sum_{k=1}^{4} \frac{\partial(v_k\mathcal{P})}{\partial y_k} - \frac{\partial}{\partial y_1}\left\langle \frac{\delta}{\delta\eta(t')}\{\delta(P(t) - y_1) \\ \times \delta(X(t) - y_2)\delta(\Pi(t) - y_3)\delta(Q(t) - y_4)\}\right\rangle_\eta. \tag{69}$$

To calculate the functional derivatives entering this equation, we notice from the direct variation of the equations of motion that:

$$\begin{bmatrix} \delta P(t)/\delta\eta(t') \\ \delta X(t)/\delta\eta(t') \\ \delta \Pi(t)/\delta\eta(t') \\ \delta Q(t)/\delta\eta(t') \end{bmatrix} = \theta(t - t')\left\{\exp\int_{t'}^{t}\mathrm{d}uA(u)\right\}_+ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{70}$$

$$A(x) = \begin{pmatrix} -\gamma/m & -\partial_{XX}^2\mathcal{H} & 0 & -\partial_{XQ}^2\mathcal{H} \\ 1/m & 0 & 0 & 0 \\ 0 & -\partial_{XQ}^2\mathcal{H} & -\gamma/m & -\partial_{XX}^2\mathcal{H} \\ 0 & 0 & 1/m & 0 \end{pmatrix}, \tag{71}$$

where $\{...\}_+$ means the chronological ordering.

Substituting the last expression into equation (62) one gets

$$\frac{\partial \mathcal{P}}{\partial t} = \sum_{k=1}^{4} \frac{\partial}{\partial y_k}\left\{v_k P + \frac{\partial}{\partial y_1}\langle \delta(P(t) - y_1)\delta(X(t) - y_2) \\ \times \delta(\Pi(t) - y_3)\delta(Q(t) - y_4)\Phi_{k1}\{x(t)\}\rangle\right\}, \tag{72}$$

where Φ is the following 4×4 matrix

$$\Phi(\{x(t)\}) = \int_0^t \mathrm{d}t'K(t')\left\{\exp\int_{t-t'}^{t}\mathrm{d}uA(x(u))\right\}_+, \tag{73}$$

and Φ_{k1} is the corresponding matrix element. This result is still exact, but intractable, since it involves the functional Φ of the history $\{X(t')\}, \{Q(t')\}$ for $t' \leq t$. In the classical limit one gets for $t' > 0$ the white noise $K(t') \rightarrow 2\gamma T\delta(t')$,

$$\Phi_{k1} = \gamma T\delta_{k1},$$

thus reproducing the corresponding classical Fokker-Planck-Kramers equation. A closed equation for $\mathcal{P}$ can be obtained also in the harmonic case, where A does not depend on $X(u), Q(u)$. These two exact realizations prompt the way to proceed in the nonlinear case. Since $K(t')$ exponentially decreases for $t > \max(h\beta, \Gamma^{-1})$, and can this time be assumed small in the quasiclassical domain, one can make a Taylor expansion of the exponent in equation (73), and keep only the first term:

$$\left\{\exp\int_{t-s}^{t}\mathrm{d}uA(x(u))\right\}_+ \approx \mathrm{e}^{sA(x(t))}. \tag{74}$$

Due to the δ-function in equation (72), we may then replace the fluctuating $x(t), q(t)$ by the sure variables x, q, after which Φ is no longer a fluctuating quantity, and can be taken outside the averaging in equation (72), thus bringing a closed equation for P. Φ will be calculated with help of the flowing formula

$$\left[e^{sA(x)}\right]_{k1} = -\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} e^{sz} \left[\frac{1}{A - z \times 1}\right]_{k1}, \quad (75)$$

where 1 is the 4×4 unit matrix. Our step (74) is still exact for the harmonic initial potential V.

The final result that we obtain is a diffusion-type equation for P itself:

$$\frac{\partial \mathcal{P}(P, X, \Pi, Q, t)}{\partial t} = -\frac{P}{m}\frac{\partial \mathcal{P}}{\partial X} + \frac{\partial}{\partial P}\left(\left[\frac{\gamma}{m}P + \frac{\partial \mathcal{H}}{\partial X}\right]\mathcal{P}\right)$$
$$- \frac{\Pi}{m}\frac{\partial \mathcal{P}}{\partial q} + \frac{\partial}{\partial \Pi}\left(\left[\frac{\gamma}{m}\Pi + \frac{\partial \mathcal{H}}{\partial Q}\right]\mathcal{P}\right)$$
$$+ \gamma D_{PP}(X, Q, t)\frac{\partial^2 \mathcal{P}}{\partial P^2} + \frac{\partial^2}{\partial P \partial X}[D_{XP}(X, Q, t)\mathcal{P}]$$
$$+ \gamma D_{\Pi P}(X, Q, t)\frac{\partial^2 \mathcal{P}}{\partial P \partial \Pi} + \frac{\partial^2}{\partial P \partial Q}[D_{QP}(X, Q, t)\mathcal{P}], \quad (76)$$

where we have changed $y_1 \mapsto P$, $y_2 \mapsto X$, $y_3 \mapsto \Pi$, $y_4 \mapsto Q$, and diffusion coefficients $D_{PP}, D_{XP}, D_{\Pi P}, D_{QP}$ are instantaneous functions of X, Q and t, and no longer functionals of the history. Since the analytic structure of the diffusion coefficients is somewhat involved, it will be explained gradually. All diffusion coefficients converge to finite values for large times. This convergence is exponential, and has a characteristic frequency

$$\min_{1 \leq k \leq 4}[\mathrm{Re}(\omega_k)], \quad (77)$$

where

$$\omega_{1,2} = \frac{\gamma}{2m}\left(1 \pm \sqrt{1 + \frac{4mb_1}{\gamma^2}}\right), \quad (78)$$

$$\omega_{3,4} = \frac{\gamma}{2m}\left(1 \pm \sqrt{1 + \frac{4mb_2}{\gamma^2}}\right), \quad (79)$$

and where

$$b_{1,2} =$$
$$- \frac{1}{2}\left[\partial_{xx}\mathcal{H} + \partial_{qq}\mathcal{H} \mp \sqrt{(\partial_{xx}\mathcal{H} - \partial_{qq}\mathcal{H})^2 + \frac{4}{m^2}[\partial_{xq}\mathcal{H}]^2}\right]. \quad (80)$$

It is seen that in order to have $\mathrm{Re}\,\omega_k \geq 0$, which is necessary for convergence, one has to require the conditions of local stability

$$\partial_{XX}\mathcal{H} + \partial_{QQ}\mathcal{H} \geq 0,$$
$$\partial_{XX}\mathcal{H}\,\partial_{XQ}\mathcal{H} \geq [\partial_{XQ}\mathcal{H}]^2. \quad (81)$$

Hereafter they will be assumed to be satisfied.

Let us now present explicit formulas for the stationary values of the diffusion coefficients:

$$D_{PP}(X, Q) = \frac{1}{m(b_1 - b_2)}\int_0^\infty \frac{\mathrm{d}\omega}{\pi}\bar{K}(\omega)\omega^2$$
$$\times \left[\frac{b_1 + \partial_{qq}\mathcal{H}}{(\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)} - \frac{b_2 + \partial_{qq}\mathcal{H}}{(\omega^2 + \omega_3^2)(\omega^2 + \omega_4^2)}\right], \quad (82)$$

$$D_{XP}(X, Q) = \frac{1}{m(b_1 - b_2)}\int_0^\infty \frac{\mathrm{d}\omega}{\pi}\bar{K}(\omega)$$
$$\times \left[\frac{(b_1 + \partial_{qq}\mathcal{H})(\omega^2 + b_1/m)}{(\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)} - \frac{(b_2 + \partial_{qq}\mathcal{H})(\omega^2 + b_2/m)}{(\omega^2 + \omega_3^2)(\omega^2 + \omega_4^2)}\right], \quad (83)$$

$$D_{\Pi P}(X, Q) = \frac{\partial_{xq}\mathcal{H}}{m(b_1 - b_2)}\int_0^\infty \frac{\mathrm{d}\omega}{\pi}\bar{K}(\omega)\omega^2$$
$$\times \left[\frac{1}{(\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)} - \frac{1}{(\omega^2 + \omega_3^2)(\omega^2 + \omega_4^2)}\right], \quad (84)$$

$$D_{PQ}(X, Q) = \frac{\partial_{xq}\mathcal{H}}{m(b_1 - b_2)}\int_0^\infty \frac{\mathrm{d}\omega}{\pi}\bar{K}(\omega)$$
$$\times \left[\frac{\omega^2 + b_1/m}{(\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)} - \frac{\omega^2 + b_2/m}{(\omega^2 + \omega_3^2)(\omega^2 + \omega_4^2)}\right], \quad (85)$$

where $\bar{K}(\omega)$ is the spectrum of $K(t)$,

$$\bar{K}(\omega) = \hbar\gamma\,\omega \coth\left(\frac{\hbar\omega\beta}{2}\right)\frac{1}{1 + (\omega/\Gamma)^2}. \quad (86)$$

Let us indicate that the diffusion process (76) is non-Markovian. In the purely classical limit the time-dependence in the diffusion coefficients disappears, and all diffusion coefficients besides $D_{PP} \to T$ disappear as well. Only then equation (76) describes a Markovian process.

5 Reduced description

Since equation (76) is still rather complicated, it is reasonable to look for relatively simple limits. One of them is the overdamped limit, which is characterized by large values of γ. In this case equation (76) can be reduced to an equation, which describes only slow variables X, Q. To proceed with this limit we shall define the following moments

$$\mathcal{M}_{kl}(X, Q, t) = \int \mathrm{d}P\,\mathrm{d}\Pi\,P^k\Pi^l\mathcal{P}(P, X, \Pi, Q, t), \quad (87)$$

and construct an equation for them starting from equation (76).

$$\dot{\mathcal{M}}_{kl} = -\frac{1}{m}\partial_X\mathcal{M}_{k+1,l} - \frac{1}{m}\partial_Q\mathcal{M}_{k,l+1}$$
$$- \frac{(k+l)\gamma}{m}\mathcal{M}_{k,l} - k\mathcal{M}_{k-1,l}\,\partial_X\mathcal{H} - l\mathcal{M}_{k,l-1}\,\partial_Q\mathcal{H}$$
$$+ \gamma k(k-1)D_{PP}\mathcal{M}_{k-2,l} - k\partial_X[D_{XP}\mathcal{M}_{k-1,l}]$$
$$- l\partial_Q[D_{QP}\mathcal{M}_{k,l-1}] + \gamma kl\,D_{\Pi P}\mathcal{M}_{k-1,l-1}. \quad (88)$$

Let us write down few first members of this hierarchy:

$$\dot{\mathcal{M}}_{00} = -\frac{1}{m}\partial_X \mathcal{M}_{10} - \frac{1}{m}\partial_Q \mathcal{M}_{01}, \tag{89}$$

$$\dot{\mathcal{M}}_{10} = -\frac{1}{m}\partial_X \mathcal{M}_{20} - \frac{1}{m}\partial_Q \mathcal{M}_{11}$$
$$- \frac{\gamma}{m}\mathcal{M}_{10} - \mathcal{M}_{00}\partial_X \mathcal{H} - \partial_X[D_{XP}M_{00}], \tag{90}$$

$$\dot{\mathcal{M}}_{01} = -\frac{1}{m}\partial_X \mathcal{M}_{11} - \frac{1}{m}\partial_Q \mathcal{M}_{02}$$
$$- \frac{\gamma}{m}\mathcal{M}_{01} - \mathcal{M}_{00}\partial_Q \mathcal{H} - \partial_Q[D_{QP}\mathcal{M}_{00}], \tag{91}$$

$$\dot{\mathcal{M}}_{20} = -\frac{1}{m}\partial_X \mathcal{M}_{30} - \frac{1}{m}\partial_Q \mathcal{M}_{21} - \frac{2\gamma}{m}\mathcal{M}_{20}$$
$$- 2\mathcal{M}_{10}\partial_X \mathcal{H} + 2\gamma D_{PP}\mathcal{M}_{00} - 2\partial_X[D_{XP}\mathcal{M}_{10}], \tag{92}$$

$$\dot{\mathcal{M}}_{02} = -\frac{1}{m}\partial_X \mathcal{M}_{12} - \frac{1}{m}\partial_Q \mathcal{M}_{03}$$
$$- \frac{2\gamma}{m}\mathcal{M}_{02} - 2\mathcal{M}_{01}\partial_Q \mathcal{H} - 2\partial_Q[D_{QP}\mathcal{M}_{01}] \tag{93}$$

$$\dot{\mathcal{M}}_{11} = -\frac{1}{m}\partial_X \mathcal{M}_{21} - \frac{1}{m}\partial_Q \mathcal{M}_{12}$$
$$- \frac{2\gamma}{m}\mathcal{M}_{11} - \mathcal{M}_{01}\partial_X \mathcal{H} - \mathcal{M}_{10}\partial_Q \mathcal{H}$$
$$+ \gamma D_{\Pi P}\mathcal{M}_{00} - \partial_X[D_{XP}\mathcal{M}_{01}] - \partial_Q[D_{QP}\mathcal{M}_{10}]. \tag{94}$$

In the first order of large γ one can skip the time-derivatives in equations (89, 90), since they are at least of order $\mathcal{O}(\gamma^{-2})$. Further from equations (89–91) one gets the following approximate relations

$$\mathcal{M}_{20} = mD_{PP}\mathcal{M}_{00}, \tag{95}$$

$$\mathcal{M}_{02} = \mathcal{O}(\gamma^{-2}), \tag{96}$$

$$\mathcal{M}_{11} = \frac{1}{2}mD_{\Pi P}\mathcal{M}_{00}. \tag{97}$$

Notice from equation (84) that $D_{\Pi P}$ is of order $1/\gamma$ for large γ. These equations are substituted into equations (89, 90), which in combination with equation (89) brings the following reduced equation for the $\mathcal{M}_{00} = \mathcal{F}(X, Q, t)$, which is the probability distribution of the slow variables:

$$\gamma \, \partial_t \mathcal{F}(X, Q, t) = \partial_X[\mathcal{F}\partial_X \mathcal{H}] + \partial_Q[\mathcal{F}\partial_Q \mathcal{H}] + \partial_{XQ}[\mathcal{F}D_{\Pi P}]$$
$$+ \partial_{XX}[\mathcal{F}D_{XX}] + \partial_{QQ}[\mathcal{F}D_{QP}], \tag{98}$$
$$D_{XX} = D_{XP} + D_{PP}. \tag{99}$$

The first two terms in the r.h.s. of equation (98) are due to drift, whereas other terms are responsible for the diffusion. Let us now discuss this situation in details. In the classical limit, which is realized for sufficiently large temperatures or for $\hbar \to 0$, one has $D_{XP} \to 0$, $D_{QP} \to 0$, $D_{\Pi P} \to 0$ and $D_{PP} \to T$. Thus, equation (98) goes to the corresponding classical Fokker-Planck equation. As equations (48, 49, 98) show, no noise is acting on the Q-particle, therefore in the classical case it just relaxes to zero and

does not fluctuate at all. Therefore, despite non-linearity of the potential $V(x)$, the classical variables X, P decouples from the quantum variables Q, Π and tend to the classical Gibbs distribution. The quantum variables disappear, as seen also from equations (96, 97). Recall that the very fact of this homogeneous disappearance is connected with the exponential damping (49) of the singular term $\hbar^2\sigma(t)^2/(8mQ^2)$ in the effective Hamiltonian (50).

On the other hand, in the quantum case equation (98) shows that both variables X and Q become correlated and involved in a common dynamics. Moreover, as follows from equation (97), there is a well-defined correlation between Π and P. In particular, through interaction with the classical variables quantum variables become coupled to the thermal bath.

6 Conclusion

This paper was devoted to the mean-field (variational) theory of quantum Brownian motion. Mean-field methods are widely applied in quantum theory [4–6] and have an established range of validity. Their general property is reduction of an initially quantum problem to an approximation involving only effective (mean-field) variables with classical (commuting) behavior [18]. The original quantum character of the problem is then reflected through an effective Hamiltonian. These properties of the mean-field description for closed systems were established also for more general cases (*e.g.* the time-dependent Hartree-Fock approximation), where the generated effective classical dynamics does not have the canonical form, but instead can be embodied into a more general Poisson structure [18].

Being motivated by the effectiveness of the mean-field approach, we considered here its application to the problem of quantum Brownian motion, which is the main representative model of quantum open systems.

Our first step was to substitute the original quantum problem by an auxiliary semi-quantum one, where only a part of degrees of freedom is quantum. In this exact step it is possible to replace the operator-valued quantum noise by an auxiliary classical Gaussian noise, which has the same spectrum as the original quantum noise. In that way we obtain the generalized Wigner-Moyal equations (21, 22). In this step our description uses negative probabilities in the sense that the unaveraged density matrix (25) does have negative eigenvalues. However, no unphysical results appear in the level of observables quantities. The negative probabilities appear as the cost for having explicitly classical elements in a quantum dynamics. This situation is reminiscent of the ordinary Wigner function which also cannot be interpreted as a probability density, but which shares some of its properties and does predict correct quantum mechanical averages.

At the second step of our description we applied the Gaussian approximation to the unaveraged Wigner function. By this procedure the initial quantum stochastic problem became reduced to a problem of two classical particles with friction and classical noise. This noise is nevertheless not white, but is correlated with the same

spectrum as the original quantum noise. Further investigation allowed us to uncover an important role played by friction in establishing classical aspects of the problem. It appeared that a singular $\hbar$-dependent term in the effective Hamiltonian (50) is diminished by friction with a characteristic time (49) inversely proportional to the damping coefficient. This ensures the existence of the unique and homogeneous classical limit for times larger than the above characteristic time. This fact is contrasting with the non-commutation of the classical ($\hbar \to 0$) and long-time $t \to \infty$ limit for (closed) Hamiltonian systems [2]. At low temperatures, where the quantum effects are essential, the dynamics of the mean-field degrees of freedom is essentially different, because they become correlated with each other. This is shown in particular by equations (97, 98). In other words, the effective classical dynamics still contains $\hbar/T$ as a correlation time of the classical noise, and therefore displays essential different dynamical behavior for high and low temperatures [15].

In the present paper we restricted ourselves to the general framework of the mean-field quantum Brownian motion. More specific applications, *e.g.* for open many-body systems, are planned to be considered in the future. Finally, it is hoped that the paper will open a road for applications of mean-field methods in quantum dissipative systems.

References

1. L.E. Ballentine, *Quantum Mechanics* (Prentice-Hall, 1990).
2. M.V. Berry, in *Chaos and Quantum Physics, Proceedings of Les Houches Summer School 1989*, edited by M.-J. Giannoni, A. Voros, J. Zinn-Justin (North Holland, Amsterdam, 1991).
3. V.I. Tatarskii, Sov. Phys. Usp. **26**, 311 (1983).
4. E. Heller, in *Chaos and Quantum Physics, Proceedings of Les Houches Summer School 1989*, edited by M.-J. Giannoni, A. Voros, J. Zinn-Justin (North Holland, Amsterdam, 1991).
5. R. Balian, M. Vénéroni, Ann. Phys. **187**, 29 (1988).
6. A.K. Pattayanak, W.C. Schieve, Phys. Rev. Lett. **72**, 2855 (1994); Phys. Rev. E **50**, 3601 (1994).
7. P. Kramer, M. Saraceno, *Geometry of Time-Dependent variational Principle in Quantum Mechanics* (Springer-Verlag, Berlin, 1981).
8. The case with completely instantaneous friction, *i.e.* $\Gamma = \infty$, is called *Ohmic* [9,10]. The term *quasi-Ohmic* comes to stress that we consider the case where Γ (which is the maximum characteristic frequency of the bath) is much larger than other characteristic frequencies of the problem, but still is finite. The same term was used in [15].
9. U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).
10. C. Gardiner, *Quantum Noise* (Springer, Berlin, 1991).
11. G.W. Ford, M. Kac, P. Mazur, J. Math. Phys. **6**, 504 (1965).
12. W. Muckenheim, Phys. Rep. **133**, 337 (1986).
13. F. Haake, R. Reibold, Phys. Rev. A **32**, 2462 (1985).
14. J.P. Paz, S. Habib, W.H. Zurek, Phys. Rev. D **47**, 488 (1993).
15. A.E. Allahverdyan, Th.M. Nieuwenhuizen, Phys. Rev. Lett. **85**, 1799 (2000); cond-mat/0011389.
16. W.V. Liu, W.C. Schieve, Phys. Rev. Lett. **78**, 3278 (1997); chao-dyn/9703012.
17. H. Grad, *Principles of the Kinetic Theory of Gases*, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, 1958).
18. R. Balian, M. Vénéroni, Ann. Phys. **195**, 324 (1988).